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SOME THEOREMS ON THE METRIC PROPERTIES OF BOOLEAN ALGEBRA

by

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Some Theorems on the Metric Properties of Boolean Algebra

Several theorems on the metric properties of Boolean Algebra have been obtained. In this report they are described with the expectation of reader's knowledge of the following report which gives the foundation of this work:

Internal Report No. 46, Metric Properties of Boolean Algebra and their Application to Switching Circuits, by D. E. Muller.

The theorems were suggested by computational results, using ingenious coding tricks which were given in the above report.

<u>THEOREM 1:</u> If there is the following distance relation between two Boolean members g_1 and g_2 in p variables, which do not include any of $x^1, x^2, \ldots, x^m$ and their complements:

$$L(g_1, g_2) = d, \qquad \text{where } p \geq m \;;$$

then the relation

$$L[g_1 f(x^1, x^2, \ldots, x^m),\ g_2 f(x^1, x^2, \ldots, x^m)] = d/2^m$$

holds, where

$$f = x^1 x^2, \ldots, x^m$$

or some of $x^1 x^2, \ldots,$ and x^m can be complemented.

<u>PROOF:</u> $$g_i = g_i \overline{x^1} + g_i x^1$$

Since g_i does not contain x^1, from the symmetrical structure of the variables, we have

$$L(g_i \overline{x^1}, 0) = L(g_i x^1, 0) \;,$$

and then

$$L(h_1 \overline{x^1}\ , g_2 \overline{x^1}) = d/2 \;,$$

and

$$L(g_1 x^1\ , g_2 x^1) = d/2 \;,$$

since

$$g_i \overline{x^1} \text{ and } g_i x^1 \text{ are disjoint.}$$

Now $g_i \overline{x^1}$ and $g_i x^1$ do not include any of $x^2, \ldots, x^m$. If we repeat the above

process on $g_i\overline{x^1}$ or $g_i x^1$, we can get the theorem for $m = 2$. Thus the repetition of the above process will produce the statement of the above theorem.

Assume that t is a member of the net of order $d = 2^{p-r}$, where p is number of independent variables. Write t in the following form:

$$t = u + v_1x^1 + v_2x^2 + v_3x^3 + w_1x^2x^3 + w_2x^1x^3 + w_3x^1x^2 + sx^1x^2x^3\ , \qquad (1)$$

where

$u = u_1 + u_2$: polynomial of $x^4, \ldots, x^p$ of the r-th degree

u_2 : polynomial of $x^4, \ldots, x^p$ of the (r-1)th degree

v_i's : polynomials of $x^4, \ldots, x^p$ of the (r-1)th degree

w_i's : polynomials of $x^4, \ldots, x^p$ of the (r-2)th degree

s : polynomials of $x^4, \ldots, x^p$ of the (r-3)th degree,

then t can be rewritten as follows:

$$\begin{aligned} t = {} & [(u + v_1 + v_2 + v_3 + w_1 + w_2 + w_3 + s)\, x^1x^2x^3 + (u + v_2 + v_3 + w_1)\, \overline{x^1}x^2x^3 \\ & + (u + v_1 + v_3 + w_2)\, x^1\overline{x^2}x^3 + (u + v_1 + v_2 + w_3)\, x^1x^2\overline{x^3}] \\ & + [(u + v_1)\, x^1\overline{x^2x^3} + (u + v_2)\, \overline{x^1}x^2\overline{x^3} + (u + v_3)\, \overline{x^1x^2}x^3 + u\, \overline{x^1x^2x^3}] \qquad (2) \end{aligned}$$

Consider the following h:

$$h = \alpha x^1x^2x^3 + \beta\overline{x^1}x^2x^3 + \gamma x^1\overline{x^2}x^3 + \delta x^1x^2\overline{x^3} \qquad (3)$$

where α, β, γ and δ are the polynomials of $x^4, \ldots, x^p$ of the (r-1)th degree. But we make the assumption that any two of α, β, γ and δ are not the same and that the sum of any two of them is not equal to any one out of the rest. Therefore:

$$\begin{aligned} (t+h) = {} & [(u + v_1 + v_2 + v_3 + w_1 + w_2 + w_3 + \alpha + s)\, x^1x^2x^3 \\ & + (u + v_2 + v_3 + w_1 + \beta)\, \overline{x^1}x^2x^3 \\ & + (u + v_1 + v_3 + w_2 + \gamma)\, x^1\overline{x^2}x^3 \\ & + (u + v_1 + v_2 + w_3 + \delta)\, x^1x^2\overline{x^3}] \\ & + [(u + v_1)\, x^1\overline{x^2x^3} + (u + v_2)\, \overline{x^1}x^2\overline{x^3} + (u + v_3)\, \overline{x^1x^2}x^3 + u\, \overline{x^1x^2x^3}] \qquad (4) \end{aligned}$$

According to whether each term in the second bracket [ ] is equal to zero or not, we can divide all the possible cases into several ones as in Table 1.

Take the case 1 where each term in the second bracket [ ] is equal to zero, that is, $u + v_1 - u + v_2 = u + v_3 = u = 0$. Then we get $u = v_1 = v_2 = v_3$, so the first term in (4) reduces to $(w_1 + w_2 + w_3 + \alpha + s)x^1x^2x^3$. From the assumption, w_i's are the polynomials of $x^4, \ldots, x^p$ of the (r-2)th degree and α is the polynomial of $x^4, \ldots, x^p$ of the (r-1)th degree. In short, α is of higher degree than w_i's. So though the terms of degree lower then (r-1) in α may vanish for some t, the terms of the (r-1)th degree do not vanish. So from Theorem 1, the first term $(w_1 + w_2 + w_3 + \alpha + s)\, x^1x^2x^3$ in (4) has the number of ones not less than $2^{p-(r-1)-3}$. Similarly the second, third and fourth terms in (4) have the number of ones not less than $2^{p-(r-1)-3}$. Totally the number of ones in $(t + h)$ in this case is not less than 2^{p-r}.

Next, take case 2 where $u + v_1 \neq 0$, and $u + v_2 = u + v_3 = u = 0$. Then we have $u = v_2 = v_3 = 0$. Now the factors of the first, third and fourth terms in (4) reduce to $v_1 + w_1 + w_2 + w_3 + \alpha + s$, $v_1 + w_2 + \gamma$ and $v_1 + w_3 + \delta$, respectively. Assume $v_1 + w_2 + \gamma = 0$. Then from $v_1 = w_2 + \gamma$, we have $v_1 + w_1 + w_2 + w_3 + \alpha + s = w_1 + w_3 + \alpha + \gamma + s$ and $v_1 + w_3 + \delta = w_2 + w_3 + \delta + \gamma$ Each of them is a polynomial in $x^4, \ldots, x^p$ of the (r-1)th degree and the terms of the (r-1)th degree can not vanish from any member t by the assumption that $\alpha \neq r$ and $\delta \neq \gamma$. So from Theorem 1, we can have at least two terms out of the first, third and fourth terms in (4), each of which has a number of ones not less than $2^{p-(r-1)-3}$. But from $v_1 \neq 0$ (from $u + v_1 \neq 0$ and $u = 0$), the fifth term, the polynomial of the (r-1)th degree has a number of ones

not less than $2^{p-(r-1)-3}$. So total number of ones is not less than 2^{p-r}.

The other cases can be treated in similar ways as follows.

In the case 3, either 1 or 4 and then either 2 or 3 is not less than $2^{p-(r-1)-3}$. In the case 4, one out of the terms 2, 3, and 4 is not less than $2^{p-(r-1)-3}$, because, if $v_2 + v_3 + w_1 + \beta = 0$ and $v_1 + v_3 + w_2 + \gamma = 0$, we have $v_1 + v_2 + w_3 + \delta = w_1 + w_2 + w_3 + \beta + \gamma + \delta$ for the fourth term, which is not zero from $\beta + \gamma \neq \delta$.

In the following three cases, we assume $u_1 = 0$ and $u_2 \neq 0$. In the case 5, either of 3 and 4 is not less than $2^{p-(r-1)-3}$. And if the third term is zero, we have $v_1 = w_2 + \gamma$. Then the first term is $v_2 + w_1 + w_3 + \alpha + \gamma + s$ and the second, $v_2 + w_1 + \beta$. So either of these two is not zero from the assumption that $\alpha + \gamma \neq \beta$. In the case 6, one out of 1, 2, and 3 is not less than $2^{p-(r-1)-3}$. If $v_2 + w_1 + \beta = 0$ and $v_1 + w_2 + \gamma = 0$, we have $v_1 + v_2 + w_1 + w_2 + w_3 + \alpha + s = w_3 + \alpha + \beta + \gamma + s$ for the first term, which is not zero from the assumption. The case 7 is obvious. In the case 8, two out of 2, 3 and 4 are not less than $2^{p-(r-1)-3}$.

If $u_1 \neq 0$, each term in equation (4) is not less than 2^{p-r-3}. So the total number of ones is not less than 2^{p-r}.

Those exhaust all possible cases.

Therefore in any case, we have $L(t + h, 0) \geq 2^{p-r}$. h itself contains a number of ones not less than 2^{p-r}. h can be rewritten as follows:

$$h = \alpha x^1 x^2 x^3 + \beta(1 + x^1)x^2 x^3 + \gamma x^1(1 + x^2)x^3 + \delta x^1 x^2 (1 + x^3) =$$

$$(\alpha + \beta + \gamma + \delta)\, x^1 x^2 x^3 + \beta x^2 x^3 + \gamma x^1 x^3 + \delta x^1 x^2.$$

As a special case, if $\alpha = \beta + \gamma + \delta$, we have $h = \beta x^2 x^3 + \gamma x^1 x^3 + \delta\, x^1 x^2$. If $\alpha \neq \beta + \gamma + \delta$, the first term in (5) can be of higher degree by one than the last three terms. If the difference between α and $\beta + \gamma + \delta$ is particularly

TABLE I

	$U+V_1$	$U+V_2$	$U+V_3$	U		THE ORDINAL NUMBER OF TERMS IN EQUATION (4): 1	2	3	4	5	6	7	8
1	0	0	0	0	$U=V_1=V_2=V_3=0$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	0	0	0	0
2	$\neq 0$ OR 0 0	0 $\neq 0$ 0	0 0 $\neq 0$	0 0 0	$U=V_2=V_3=0$	$V_1+W_1+W_2+W_3+\alpha$?	$W_1+\beta$ $\geq 2^{p-(r-1)-3}$	$V_1+W_2+\gamma$?	$V_1+W_3+\delta$?	$\geq 2^{p-(r-1)-3}$	0	0	0
3	$\neq 0$ OR $\neq 0$ 0	$\neq 0$ 0 $\neq 0$	0 $\neq 0$ $\neq 0$	0 0 0	$U=V_3=0$	$V_1+V_2+W_1+W_2+W_3+\alpha$?	$V_2+W_1+\beta$?	$V_1+W_2+\gamma$?	$V_1+V_2+W_3+\delta$?	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	0	0
4	$\neq 0$	$\neq 0$	$\neq 0$	0	$U=0$	$V_1+V_2+V_3+W_1+W_2+W_3+\alpha$	$V_2+V_3+W_1+\beta$?	$V_1+V_3+W_2+\gamma$?	$V_1+V_2+W_3+\delta$?	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	0
5	$\neq 0$ OR 0 0	0 $\neq 0$ 0	0 0 $\neq 0$	$U_1=0, U_2\neq 0$ $U_1=0, U_2\neq 0$ $U_1=0, U_2\neq 0$	$U_2=V_2=V_3$	$V_1+V_2+W_1+W_2+W_3+\alpha$?	$V_2+W_1+\beta$?	$V_1+W_2+\gamma$?	$V_1+W_3+\delta$?	$\geq 2^{p-(r-1)-3}$	0	0	$\geq 2^{p-(r-1)-3}$
6	$\neq 0$ OR $\neq 0$ 0	$\neq 0$ 0 $\neq 0$	0 $\neq 0$ $\neq 0$	$U_1=0, U_2\neq 0$ $U_1=0, U_2\neq 0$ $U_1=0, U_2\neq 0$	$U_2=V_3$	$V_1+V_2+W_1+W_2+W_3+\alpha$?	$V_2+W_1+\beta$?	$V_1+W_2+\gamma$?	$V_1+V_2+V_3+W_3+\delta$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	0	$\geq 2^{p-(r-1)-3}$
7	$\neq 0$	$\neq 0$	$\neq 0$	$U_1=0, U_2\neq 0$						$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$	$\geq 2^{p-(r-1)-3}$
8	0	0	0	$U_1=0, U_2\neq 0$	$U_2=V_1=V_2=V_3$	$W_1+W_2+W_3+\alpha$ $\geq 2^{p-(r-1)-3}$	$V_1+W_1+\beta$?	$V_1+W_2+\gamma$?	$V_1+W_3+\delta$?	0	0	0	$\geq 2^{p-(r-1)-3}$

a constant, h has a form:

$$h = x^1x^2x^3 + \beta x^2x^3 + \gamma x^1x^3 + \delta\, x^1x^2 \,.$$

As a conclusion, we can get the following theorem.

<u>THEOREM 2:</u> The Boolean member of the following form:

$$h = (\,\alpha + \beta + \gamma + \delta)\, x^1x^2x^3 + \beta x^2x^3 + \gamma x^1x^2 + \delta x^1x^2 \qquad (6)$$

where α, β, γ and δ are the polynomials of the (r-1)th degree [not less than (r-1)th degree], has the minimum distance $d = 2^{p-r}$ from any member of the net of order $d = 2^{p-r}$, if any two out of α, β, γ and δ are not the same and then sum of any two out of them is not equal to any one of the rest.

Using the expansion (2), we can get the following corollary.

COROLLARY 1: If the relation

$$L(t,\, h') \geq d = 2^{p-r} \qquad (7)$$

holds for any member t of order $d = 2^{p-r}$, where h' is the polynomial of degree not less than r + 1 and does not include any variable of x^1, x^2 and x^3, then we have

$$L(t + h,\, h') \geq d$$

for any member t, where h is given by (6) in Theorem 2.

<u>PROOF:</u> From (4) and the fact that h' does not include any of x^1, x^2 and x^3, we get

$$t + h + h' = [(h' + u + v_1 + v_2 + v_3 + w_1 + w_2 + w_3 + \alpha + s)x^1x^2x^3 + (h' + u \; v_2 + v_3 + w_1 + \beta\,)\, \overline{x^1}x^2x^3 + (h' + u + v_1 + v_2 + w_3 + \delta)\, x^1x^2\overline{x^3}] + [(h' + u + v_1)\, x^1\overline{x^2x^3} + (h' + u \,'\, v_2\, \overline{x^1}x^2\overline{x^3} + (h' + u + v_3)\overline{x^1x^2}x^3 + (h' + u)\overline{x^1x^2x^3}]. \qquad (9)$$

Take the factor $(h' + u + v_1 + v_2 + v_3 + w_1 + w_2 + w_3 + \alpha\,)$ **in the first term** in (9). The whole terms from the second up to the last α can be expressed as a member of the net of order $d = 2^{p-r}$, since all of them are polynomials of degree not more than r.

So from Theorem 1:

$$L(h' + u + v_1 + v_2 + v_3 + w_1 + w_2 + w_3 + \propto + s)x_1x_2x_3, 0) \geq d/2^3.$$

Similarly, any term in (9) has a number of ones not less than $d/2^3$. So totally the relation (8) is satisfied.

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